

MATRIX THEORY OF pp WAVES*

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The Matrix Theory that has been proposed for various pp wave backgrounds is discussed. Particular emphasis is on the existence of novel nontrivial supersymmetric solutions of the Matrix Theory. These correspond to branes of various shapes (ellipsoidal, paraboloidal, and possibly hyperboloidal) that are unexpected from previous studies of branes in pp wave geometries.

1. INTRODUCTION

The BMN¹ Matrix Theory, which describes the maximally supersymmetric pp wave of M-theory², is a very nice arena for learning about M-theory. It is simple enough to be reasonably tractable, yet describes a curved background. Moreover, it has a dimensionless parameter $\frac{\mu\alpha'}{R}$ —the ratio, in string units, between the strength of the four-form flux and the DLCQ radius—which, when large, permits a perturbative treatment of the Yang-Mills theory³. Additionally, the four-form flux supports fuzzy sphere solutions to the Matrix Theory¹. The fuzzy spheres are the general vacua of the Matrix Theory, which preserve *all* the SUSYs. These fuzzy spheres have very interesting properties upon compactification⁴ of the theory to a IIA Matrix String Theory. These results have already appeared in the literature^{5,6} and space constraints prevent us from summarizing them here.

There are many pp wave solutions to M-theory^{7,8,9,10} and a Matrix Theory has been proposed^{8,11} for each. The original argument noted that the form reproduces BMN and has the right number of SUSYs⁸. A derivation from membrane quantization has since been given¹¹. One can also show, beyond counting fermionic generators, that the full SUSY algebras match¹².

*To appear in the Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3) 10 – 14 Sept 2003 — © World Scientific.

Having proposed a Matrix Theory for each M-theory pp wave, one should now study them. The zeroth question to answer is: what are the vacua which preserve all of the SUSYs? We will study this for three pp waves: the Penrose limit of $\text{AdS}_3 \times S^3$ ^{7,8,13,14}, the T-dual⁴ of the maximally supersymmetric pp wave of the IIB theory¹⁵; and the 26 supercharge pp wave¹⁰. For the first two, we find, at infinite N , brane solutions which do not follow from treating g_{++} as a superpotential¹⁶.^a

2. MATRIX THEORIES

The general 11-dimensional pp wave considered is^{17,8,9}

$$ds^2 = 2dx^+ dx^- - \left[\sum_{i=1}^9 \mu_i^2 (x^i)^2 \right] (dx^+)^2 + (dx^i)^2, \quad F = dx^+ \wedge \Theta, \quad (1)$$

with constant μ_i , Θ , and also Θ is a “spatial” (support in the x^i -directions)^b three-form, satisfying the equation of motion (e.o.m.) $\Theta_{ijk}^2 = 12 \sum_i \mu_i^2$.

This geometry admits $16 + 2n$ Killing spinors. To present them—and define n —set, (with all sums explicit here, and $U_{(i)} \equiv 3\Gamma^i \Theta \Gamma^i + \Theta$)

$$\Omega_+ \equiv -\frac{1}{12} \Theta (\Gamma^+ \Gamma^- + \mathbb{1}), \quad \Omega_- \equiv 0, \quad \Omega_i \equiv \frac{1}{24} \Gamma^i U_{(i)} \Gamma^+. \quad (2)$$

The solutions of the Killing spinor equation, $\mathcal{D}_A \epsilon = \nabla_A \epsilon - \Omega_A \epsilon = 0$, are

$$\epsilon(x, \epsilon_0) = \left[1 + \sum_i x^i \Omega_i \right] e^{-\frac{1}{12} (\Gamma^+ \Gamma^- + \mathbb{1}) \not{x}^+} \epsilon_0, \quad (3)$$

where the constant spinor ϵ_0 obeys^{17,8,9}

$$U_{(i)}^2 \Gamma^+ \epsilon_0 = -144 \mu_i^2 \Gamma^+ \epsilon_0. \quad (4)$$

So there are 16 “standard” SUSYs which obey $\Gamma^+ \epsilon_0 = 0$ and an even number, $2n$, of “supernumerary”⁷ SUSYs obeying

$$U_{(i)}^2 \epsilon_0 = -144 \mu_i^2 \epsilon_0, \quad (5)$$

for each i . Equation (5) need not have any solutions, and never has 12 or 14 solutions¹⁰. Existence of “supernumerary” SUSYs guarantees a solution to the e.o.m.s, by summing Eq. (5) over i and tracing over Killing spinors.

^aUnfortunately there is no room to list the myriad papers on branes in pp waves.

^bThe index i, j, \dots runs from 1 to 9, and A, B, \dots runs over all coordinates. Γ -matrices are defined with respect to the elfbein $e^- = dx^- - \frac{1}{2} \sum_{i=1}^9 \mu_i^2 (x^i)^2 dx^+$, $e^+ = dx^+$, $e^i = dx^i$. The Feynman slash, e.g. $\not{\Theta} = \frac{1}{3!} \Gamma^{ijk} \Theta_{ijk}$, is used extensively.

In units for which the 11-dimensional Planck length, $\ell_P = 1$, and employing Majorana $\mathfrak{so}(9)$ spinors Ψ , the Matrix Theory action^{8,11} is^c

$$S = R \int d\tau \text{Tr} \left\{ \frac{1}{2R^2} (D_\tau X^i)^2 + \frac{i}{R} \Psi^\top D_\tau \Psi + \Psi^\top \Gamma^i [X^i, \Psi] + \frac{1}{4} [X^i, X^j]^2 - \frac{1}{2} \sum_i \frac{\mu_i^2}{R^2} (X^i)^2 - \frac{i}{4R} \Psi^\top \Theta \Psi - \frac{i}{3R} \Theta_{ijk} X^i X^j X^k \right\}. \quad (6)$$

The first line is the familiar BFSS Matrix Theory¹⁸ action; the second line contains mass terms for the bosons and fermions, and the Myers term^{19,20}.

The Matrix Theory (6) is invariant under the nonlinearly realized SUSY $\delta\Psi = e^{\frac{1}{4}\Theta\tau}\epsilon_+$ corresponding to the 16 “standard” SUSYs of the pp wave. Consider also the transformations,

$$\delta X^i = i\Psi^\top \Gamma^i \epsilon(\tau), \quad \delta A_\tau = iR\Psi^\top \epsilon(\tau), \quad \epsilon(\tau) \equiv e^{-\frac{1}{12}\Theta\tau}\epsilon_0, \quad (7a)$$

$$\delta\Psi = \frac{1}{2R} D_\tau X^i \Gamma^i \epsilon(\tau) + \frac{1}{R} X^i \hat{\Omega}_i \epsilon(\tau) + \frac{i}{4} [X^i, X^j] \Gamma^{ij} \epsilon(\tau), \quad (7b)$$

where $\hat{\Omega}_i \equiv \frac{1}{24} \Gamma^i U_{(i)}$ and ϵ_0 is constant (*cf.* Bonelli²¹). These transformations preserve the action precisely when ϵ_0 obeys (5). So the Matrix quantum mechanics (6) preserves exactly the right number of SUSYs to describe the pp wave (1)^{7,11}. Indeed, the SUSY algebras match¹².

3. THE $\text{AdS}_3 \times S^3$ pp WAVE

The Penrose limit of $\text{AdS}_3 \times S^3$ has been considered in^{7,8,13,14}. Lifting the geometry supported with NS-NS flux to 11-dimensions,

$$ds^2 = 2dx^+ dx^- - \mu^2 \sum_{a=1}^4 (x^a)^2 (dx^+)^2 + \sum_{i=1}^9 (dx^i)^2, \quad (8)$$

$${}^{(4)}F = 2\mu dx^+ \wedge dx^1 \wedge dx^2 \wedge dx^9 + 2\mu dx^+ \wedge dx^3 \wedge dx^4 \wedge dx^9.$$

The M-theory direction is x^9 , and the directions x^a , $a=1 \cdots 4$ are the space-like $\text{AdS}_3 \times S^3$ directions transverse to the null geodesic of the Penrose limit.

Vanishing of the fermionic variation (7b) gives the conditions

$$\begin{aligned} [X^1, X^2] &= [X^3, X^4], \quad [X^2, X^3] = [X^1, X^4], \quad [X^1, X^3] = -[X^2, X^4], \\ [X^2, X^9] &= i\frac{\mu}{R} X^1, \quad [X^1, X^9] = -i\frac{\mu}{R} X^2, \\ [X^4, X^9] &= i\frac{\mu}{R} X^3, \quad [X^3, X^9] = -i\frac{\mu}{R} X^4. \end{aligned} \quad (9)$$

^c $D_\tau X^i = \partial_\tau X^i + i[A_\tau, X^i]$ and $(\Psi_\alpha \Psi_\beta)^* = \Psi_\beta^* \Psi_\alpha^*$. R is the DLCQ radius.

All other commutators vanish, and the vacua are static.

For finite N , all vacua preserving all eight linearly-realized SUSYs are trivial: $X^1 = X^2 = X^3 = X^4 = 0$. However, there is a family of solutions for infinite N . Specifically, take X^1, X^2 and X^3, X^4 to form two noncommutative planes, with equal noncommutativity parameter,

$$[X^1, X^2] = i\vartheta = [X^3, X^4]; \quad [X^1, X^3] = 0 = [X^2, X^3], \quad (10)$$

and take

$$X^9 = X_0^9 - \frac{\mu}{2R\vartheta} [(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2], \quad (11)$$

where X_0^9 commutes with all the matrices. Then it is straightforward to see that equations (9) are satisfied. This solution describes a longitudinal fivebrane of a (fuzzy) paraboloidal shape (11). The fivebrane wraps the entire “ $AdS_3 \times S^3$ directions” but it extends into the M-theory direction through the paraboloid equation.

4. THE 26 SUPERCHARGE MATRIX THEORY VACUA

The pp wave with 26 supercharges was presented in¹⁰. Supersymmetric vacua obey $(I, J, \dots = 1 \dots 7, I'', J'', \dots = 1, 2, 3)$, with the conventions¹⁰)

$$\begin{aligned} D_\tau X^I &= 0, I = 1 \dots 7, \quad D_\tau X^8 = \frac{\mu}{2} X^9, \quad D_\tau X^9 = -\frac{\mu}{2} X^8, \\ X^4 &= X^5 = X^6 = X^7 = 0, \quad [X^{I''}, X^{J''}] = -i \frac{\mu}{R} \epsilon_{I'' J'' K''} X^{K''}, \\ [X^8, X^{I''}] &= 0 = [X^9, X^{I''}]. \end{aligned} \quad (12)$$

That is, states preserving all the SUSYs are fuzzy spheres at the origin of the $(4, 5, 6, 7)$ -hyperplane, and orbit in the $(8, 9)$ -plane with frequency $\frac{\mu}{2}$.

Recall¹⁰ that the 26 supercharge pp wave can be compactified to a 26 supercharge IIA pp wave. Such a reduction occurs along a Killing vector which rotates in the $(8, 9)$ -plane. Thus, in the IIA theory, the fully supersymmetric solutions (12) are static. This is reminiscent of the maximally supersymmetric pp wave, for which⁵ the orbiting fuzzy spheres broke the same half of the (supernumerary) SUSYs as were broken by the reduction to the IIA theory, yielding fully (24 supercharge) supersymmetric IIA fuzzy spheres located at any static value of X_{IIA}^8 . Here, the solutions have identical form, but with two additional supercharges.

5. IIB T-dual Matrix Theory Vacua

The T-dual, lifted to M-theory, of the IIB maximally supersymmetric pp wave is⁴ (the coordinates have been reindexed relative to the reference)

$$ds^2 = 2dx^+ dx^- - \left[4\mu^2 \sum_{I=1}^6 (x^I)^2 + 16\mu^2 (x^7)^2 \right] (dx^+)^2 + \sum_{i=1}^9 (dx^i)^2, \quad (13a)$$

$$F = -4\mu dx^+ \wedge dx^7 \wedge dx^8 \wedge dx^9 + 8\mu dx^+ \wedge dx^5 \wedge dx^6 \wedge dx^7. \quad (13b)$$

The linear SUSYs of the Matrix Theory are parametrized by $\Gamma^{5689}\epsilon_0 = \epsilon_0$.

Fully supersymmetric vacua must be static, at $X^{1\dots 4} = 0$ and satisfy

$$-2\frac{\mu}{R}X^7 + \frac{i}{2}[X^8, X^9] - \frac{i}{2}[X^5, X^6] = 0, \quad (14a)$$

$$[X^6, X^7] = 2i\frac{\mu}{R}X^5, \quad [X^7, X^5] = 2i\frac{\mu}{R}X^6, \quad (14b)$$

$$[X^5, X^8] = -[X^6, X^9], \quad [X^6, X^8] = [X^5, X^9], \quad (14c)$$

$[X^7, X^8] = 0 = [X^7, X^9]$. These equations are similar to those for non-maximally SUSic vacua of the BMN theory²². Solutions to equation (14) satisfy the e.o.m.s.

For finite N , the only solutions are fuzzy ellipsoids, $([J^a, J^b] = i\epsilon_{abc}J^c)$

$$X^5 = 2\sqrt{2}\frac{\mu}{R}J^1, \quad X^6 = 2\sqrt{2}\frac{\mu}{R}J^2, \quad X^7 = 2\frac{\mu}{R}J^3. \quad (15)$$

Constant values of $X^{8,9}$, that are diagonal and proportional to the identity in each irreducible $SU(2)$ block, give the positions of the fuzzy ellipsoids.

At strictly infinite N there are many more solutions. For example,

$$X^{5,6} = 2\sqrt{2}\frac{\mu}{R}J^{1,2} \otimes \mathbb{1}, \quad X^7 = 2\frac{\mu}{R}\left(J^3 - \frac{1}{8}\vartheta\mathbb{1}\right) \otimes \mathbb{1}, \quad (16a)$$

$$X^8 = \mathbb{1} \otimes \hat{x}^8, \quad X^9 = \mathbb{1} \otimes \hat{x}^9, \quad (16b)$$

where $[\hat{x}^8, \hat{x}^9] = i\vartheta$ is a c-number. That is, $X^{5,6,7}$ form a fuzzy ellipsoid and X^8 and X^9 parameterize a noncommutative plane orthogonal to the ellipsoid and translated, along X^7 , by an amount $-\frac{\mu}{4R}\vartheta$ from the origin. This is the longitudinal M5-brane, formed as a stack of M2-branes^{18,23}, with one M2-brane blown up into a fuzzy ellipsoid. Thus, the solution (16) describes a longitudinal M5-brane of topology $\mathbb{R}^{1,3} \times S^2$.

One can also find a solution for every complex simple Lie algebra of rank 2. If the algebra is not simply-laced—*i.e.* except for $\mathfrak{su}(3)$ —the algebra gives two inequivalent solutions. Explicitly, if the two roots of the

algebra are $\alpha_1 = (a_1, -b_1)$ and $\alpha_2 = (0, a_2)$, with Cartan subalgebra h_1, h_2 , ($[h_1, \alpha_1] = a_1 \alpha_1$, $[\alpha_2, \alpha_2^\dagger] = a_2 h_2$, etc.), then

$$\begin{aligned} X^7 &= \frac{2\mu}{a_1 R} h_1, & X^5 &= \frac{2\mu}{a_1 R} (e_1 + e_1^\dagger), & X^6 &= \frac{2\mu}{a_1 R i} (e_1 - e_1^\dagger), \\ X^8 &= -2i \sqrt{\frac{b_1}{a_1^2 a_2}} \frac{\mu}{R} (e_2 + e_2^\dagger), & X^9 &= -2 \sqrt{\frac{b_1}{a_1^2 a_2}} \frac{\mu}{R} (e_2 - e_2^\dagger), \end{aligned} \quad (17)$$

is a solution. Demanding hermitian matrices then restricts to a particular noncompact realification of each algebra. The resulting realifications are $\mathfrak{su}(2, 1)$, $\mathfrak{sp}(1, 1) \cong \mathfrak{so}(1, 4)$, $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3)$ and $\mathfrak{g}_{2(2)}$. Two inequivalent solutions are obtained from the last algebra. As these are all noncompact, their nontrivial unitary representations (hermitian matrices) are all infinite dimensional.

For example, explicit²⁴ $\mathfrak{su}(2, 1)$ solutions can be written in terms of three commuting sets of annihilation operators a_1, a_2, b , and their hermitian conjugate creation operators,

$$\begin{aligned} X^5 &= 2 \sqrt{\frac{2}{3}} \frac{\mu}{R} (a_1^\dagger a_2 + a_2^\dagger a_1), & X^6 &= -i 2 \sqrt{\frac{2}{3}} \frac{\mu}{R} (a_1^\dagger a_2 - a_2^\dagger a_1), \\ X^8 &= \frac{2}{\sqrt{3}} \frac{\mu}{R} (a_1 b + a_1^\dagger b^\dagger), & X^9 &= -i \frac{2}{\sqrt{3}} \frac{\mu}{R} (a_1 b - a_1^\dagger b^\dagger), \\ X^7 &= \frac{2}{3} \frac{\mu}{R} (a_1^\dagger a_1 - 2 a_2^\dagger a_2 - b b^\dagger). \end{aligned} \quad (18)$$

The quadratic Casimir of $\mathfrak{su}(2, 1)$ takes the form

$$\begin{aligned} C_2^{\mathfrak{su}(2,1)} &= -\frac{3}{4} \frac{R^2}{\mu^2} ((X^8)^2 + (X^9)^2) + \frac{3}{8} \frac{R^2}{\mu^2} ((X^5)^2 + (X^6)^2 + 2(X^7)^2) \\ &\quad - \frac{9}{64} \frac{R^4}{\mu^4} [X^8, X^9]^2 + \frac{9}{32} \frac{R^4}{\mu^4} ([X^5, X^8]^2 + [X^5, X^9]^2). \end{aligned} \quad (19)$$

The first line suggests a hyperboloidal interpretation to the solution, though this needs refinement.

Compactifying the Matrix Theory along X^8 gives¹², at least for weak string coupling, the Green-Schwarz action for the IIB string in the pp wave background. The radius of compactification matches precisely. This demonstration requires a field redefinition in the Matrix String Theory that is the inverse of the coordinate transformation that makes the isometry manifest⁴ on the IIB side. Unfortunately, there is no space to give details here.

Acknowledgments

This work was supported in part by Department of Energy contract #DE-FG01-00ER45832.

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